

Ordered Spaces, Metric Preimages, and Function Algebras ^{*}

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Abstract

We consider the *Complex Stone-Weierstrass Property* (CSWP), which is the complex version of the Stone-Weierstrass Theorem. If X is a compact subspace of a product of three linearly ordered spaces, then X has the CSWP if and only if X has no subspace homeomorphic to the Cantor set. In addition, every finite power of the double arrow space has the CSWP. These results are proved using some results about those compact Hausdorff spaces which have scattered-to-one maps onto compact metric spaces.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. As usual, a subset of a space is *perfect* iff it is closed and non-empty and has no isolated points, so X is *scattered* iff X has no perfect subsets.

The usual version of the *Stone-Weierstrass Theorem* involves subalgebras of $C(X, \mathbb{R})$, and is true for all compact X . If one replaces the real numbers \mathbb{R} by the complex numbers \mathbb{C} , the “theorem” is true for some X and false for others, so it becomes a *property* of X :

Definition 1.1 *If X is compact, then $C(X) = C(X, \mathbb{C})$ is the algebra of continuous complex-valued functions on X , with the usual supremum norm. $\mathcal{A} \subseteq C(X)$ means that \mathcal{A} is a subalgebra of $C(X)$ which separates points and contains the constant functions. $\mathcal{A} \sqsubseteq C(X)$ means that $\mathcal{A} \subseteq C(X)$ and \mathcal{A} is closed in $C(X)$.*

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X has the Complex Stone-Weierstrass Property (CSWP) iff every $\mathcal{A} \subseteq C(X)$ is dense in $C(X)$; equivalently, iff every $\mathcal{A} \subseteq C(X)$ equals $C(X)$.

The CSWP is easily seen to be true for finite spaces. The complex analysis developed in the 1800s shows that the CSWP is false for many compact subspaces of the plane; for example, it is false for the unit circle \mathbb{T} ; the classic counter-example being the algebra of complex polynomials $\mathcal{P} \subseteq C(\mathbb{T})$. These remarks are subsumed by results of W. Rudin [14, 15] from the 1950s:

Theorem 1.2 *Let X be any compact space.*

1. *If X contains a copy of the Cantor set, then X fails the CSWP.*
2. *If X is scattered, then X satisfies the CSWP.*

If a compact space is metrizable (equivalently, second countable), then it contains a Cantor subset iff it is not scattered, so as Rudin pointed out:

Corollary 1.3 *If X is compact metric, then X satisfies the CSWP iff X does not contain a copy of the Cantor set.*

One might conjecture that this corollary holds for all compact X , but that was refuted in 1960 by Hoffman and Singer [9] (see also [4, 8]); their results imply that any compactum containing $\beta\mathbb{N}$ fails the CSWP.

However, the corollary does hold for some more “reasonable” classes of spaces. Kunen [11] showed in 2004:

Theorem 1.4 *If X is a compact LOTS, then X satisfies the CSWP iff X does not contain a copy of the Cantor set.*

As usual, a LOTS is a linearly ordered topological space. Of course, the \rightarrow of this result is clear from Theorem 1.2; only the \leftarrow was new. This theorem shows that there are some non-scattered spaces with the CSWP, such as the double arrow space of Alexandroff and Urysohn (see Definition 2.1, or [1], p. 76).

One can now ask whether there are further classes of “reasonable” spaces for which results such as Corollary 1.3 and Theorem 1.4 hold. We do not know the best possible result along this line, but we shall prove in Section 5:

Theorem 1.5 *If X is compact and $X \subseteq L_0 \times L_1 \times L_2$, where L_0, L_1, L_2 are LOTSes, then X has the CSWP iff X does not contain a copy of the Cantor set.*

Here, we may assume that L_0, L_1, L_2 are compact (otherwise, replace them by the projections of X). It is unknown whether the product of two spaces with the CSWP must also have the CSWP. Even if this turns out to be true, Theorem 1.5 is not immediate from Theorem 1.4, since X is an arbitrary compact subset of the product, and L_0, L_1, L_2 may fail the CSWP (i.e., have Cantor subsets).

By a slightly different argument, we shall show in Section 7:

Theorem 1.6 *If L is the double arrow space, then L^n has the CSWP for every finite n .*

Theorems 1.5 and 1.6 are proved using some results from Section 3 about spaces which have scattered-to-one maps onto metric spaces. In Theorem 1.6, there is a natural $f : L^n \rightarrow [0, 1]^n$ for which the inverse of each point is scattered (and of size 2^n). In Theorem 1.5, the L_j need not have any scattered-to-one maps onto metric spaces, but a standard argument using measures reduces the proof of Theorem 1.5 to the case where the L_j are separable (see Section 4), in which case X must have an eight-to-one map onto a compact metric space.

If L_0, L_1, L_2 are separable in Theorem 1.5, then X must also be first countable, and hence “small” in the cardinal functions sense (see Juhász [10]). However, we do not believe that there is a notion of “reasonable” involving only cardinal functions. In [6] it is shown that in some models of set theory, there is a compact X which does not contain Cantor subsets and which fails the CSWP, such that X is both hereditarily separable and hereditarily Lindelöf (and hence also first countable). In these models, $2^{\aleph_0} = \aleph_1$ and the standard cardinal functions of our X (all either \aleph_0 or \aleph_1) are the least possible among non-metric compacta.

Section 2 reviews some elementary fact about LOTSeS. Section 6 discusses the notion of a *removable space* defined in [5]; this is a strengthening of the CSWP used in Section 7.

Definition 1.7 *Let \mathfrak{K} be a class of compact spaces. \mathfrak{K} is closed-hereditary iff every closed subspace of a space in \mathfrak{K} is also in \mathfrak{K} . \mathfrak{K} is local iff \mathfrak{K} is closed-hereditary and for every compact X : if X is covered by open sets whose closures lie in \mathfrak{K} , then $X \in \mathfrak{K}$.*

Classes of compacta which restrict cardinal functions (first countable, second countable, countable tightness, etc.) are clearly local, whereas the class of compacta which are homeomorphic to a LOTS is closed-hereditary, but not local.

It is easily seen that the CSWP is closed-hereditary; this is Lemma 1.3 of [11], but the proof is implicit in Rudin [14]. Thus, to prove part (1) of Theorem 1.2 in [14], it was sufficient to show that the Cantor set itself fails the CSWP.

The removable spaces form a local class (see Section 6). It is unknown whether the CSWP is a local property. A proof that it is local cannot be completely trivial.

For example, locality would imply that the failure of the CSWP for \mathbb{T} yields the failure of the CSWP for an arc $A \subseteq \mathbb{T}$. Now, A does in fact fail the CSWP, since it contains a Cantor set, but we do not know how to construct a counter-example on A directly from the polynomial algebra $\mathcal{P} \subseteq C(\mathbb{T})$; note that the restriction $\mathcal{P}|_A \subseteq C(A)$ is dense in $C(A)$ by Mergelyan's Theorem.

2 Ordered Spaces

We begin by defining the double arrow space and some variants thereof:

Definition 2.1 *$I = [0, 1]$. If $\Lambda : I \rightarrow \omega$, then $I_\Lambda = \bigcup_{x \in I} \{x\} \times \{0, 1, \dots, \Lambda(x)\}$, which is given the lexicographic order and the usual order topology. If $S \subseteq (0, 1)$, then $I_S = I_{\chi_S}$, where χ_S is the characteristic function; then for $x \in S$, let $x^- = (x, 0)$ and $x^+ = (x, 1)$; while if $x \notin S$, let $x^- = x^+ = (x, 0)$. The double arrow space is $I_{(0,1)}$. For any Λ , the map $(x, \ell) \mapsto x$ is the standard map from I_Λ onto I .*

So, we form I_Λ by splitting each $x \in S$ into $\Lambda(x) + 1$ neighboring points. For I_S , we split each $x \in S$ into two neighboring points, x^-, x^+ , and we don't split the points in $I \setminus S$; it is convenient to have x^\pm defined for *all* $x \in I$, so, for example, we can say that for all $a < b$ in I , (a^+, b^-) is an open interval in I_S . I_S has no isolated points because $0, 1 \notin S$. The double arrow space is obtained by splitting all points other than $0, 1$. $I_\emptyset \cong I$, and $I_{\mathbb{Q} \cap (0,1)}$ is homeomorphic to the Cantor set.

Lemma 2.2 *For each $S \subseteq (0, 1)$, I_S is a compact separable LOTS with no isolated points. I_S is second countable iff S is countable. Every I_Λ is a compact first countable LOTS.*

I_Λ will not be separable unless $\{x : \Lambda(x) > 1\}$ is countable. The study of compact separable LOTSes can be reduced to spaces of the form I_S . First note, by Lutzer and Bennett [13]:

Lemma 2.3 *If X is a separable LOTS, then X is hereditarily separable and hereditarily Lindelöf.*

Also, it is easy to check:

Lemma 2.4 *If X is a LOTS and H is a compact subset of X , then the relative topology and the order topology agree on H .*

Relating this to our I_S :

Lemma 2.5 *Let X be a compact separable LOTS. Then*

1. *If X is perfect, then X is homeomorphic to I_S for some $S \subseteq (0, 1)$.*
2. *If X is not second countable, then X has a closed subspace which is homeomorphic to I_S for some uncountable $S \subseteq (0, 1)$.*
3. *X is homeomorphic to a subset of I_S for some $S \subseteq (0, 1)$.*

Proof. For (1): Let $E \subseteq X$ be countable and dense in X and contain the first and last elements of X . Let B be the set of all $b \in E$ such that for some $a \in E$: $a < b$ and $(a, b) = \emptyset$. Let $D = E \setminus B$. Since X has no isolated points, D is also dense in X and contains the first and last elements of X , and is also densely ordered. Let f be an order isomorphism from D onto $\mathbb{Q} \cap [0, 1]$. Then f extends in a natural way to a continuous $F : X \rightarrow [0, 1]$, and $1 \leq |F^{-1}\{r\}| \leq 2$ for each $r \in [0, 1]$. Let $S = \{r : |F^{-1}\{r\}| = 2\}$.

For (2): Since X is hereditarily Lindelöf, the Cantor-Bendixson sequence of X has countable length and removes countably many points. Thus, X is not scattered, and, letting H be the perfect kernel of X , $X \setminus H$ is countable. Then H is separable and not second countable, so $H \cong I_S$ for some uncountable S .

For (3): Apply (1) to the space obtained from X by replacing each isolated point by a copy of the double arrow space. ☕

Note that $(I_S)^2$ is separable, but it is not hereditarily separable when S is uncountable; in fact, more general I_Λ occur naturally in such products. Fixing an uncountable $S \subseteq (0, 1)$, let $L_n = I_{\Lambda_n}$, where $\Lambda_n(x) = n$ for $n \in S$, and $\Lambda_n(x) = 0$ for $n \notin S$. Then L_n is not separable whenever $n \geq 2$, and the diagonal of $(I_S)^k$ is homeomorphic to L_{2^k-1} .

3 Tight Maps and Dissipated Spaces

We recall some definitions and results from [12]. As usual, $f : X \rightarrow Y$ means that f is a *continuous* map from X to Y , and $f : X \twoheadrightarrow Y$ means that f is a continuous map from X *onto* Y .

Definition 3.1 *Assume that X, Y are compact and $f : X \rightarrow Y$.*

- ☞ *A loose family for f is a disjoint family \mathcal{P} of closed subsets of X such that for some non-scattered $Q \subseteq Y$, $Q = f(P)$ for all $P \in \mathcal{P}$.*
- ☞ *f is κ -tight iff there are no loose families for f of size κ .*
- ☞ *f is tight iff f is 2-tight.*

This notion gets weaker as κ gets bigger. f is 1-tight iff $f(X)$ is scattered, so that “2-tight” is the first non-trivial case. f is trivially $|X|^+$ -tight. The usual projection from $[0, 1]^2$ onto $[0, 1]$ is not 2^{\aleph_0} -tight.

Some easy equivalents to “ κ -tight” are described in Lemma 2.2 of [12]:

Lemma 3.2 *Assume that X, Y are compact and $f : X \rightarrow Y$. Then (1) \leftrightarrow (2). If κ is finite and Y is metric, then all four of the following are equivalent:*

1. *There is a loose family of size κ .*
2. *There is a disjoint family \mathcal{P} of perfect subsets of X with $|\mathcal{P}| = \kappa$ and a perfect $Q \subseteq Y$ such that $Q = f(P)$ for all $P \in \mathcal{P}$.*
3. *For some metric M and $\varphi \in C(X, M)$, $\{y \in Y : |\varphi(f^{-1}\{y\})| \geq \kappa\}$ is uncountable.*
4. *Statement (3), with $M = [0, 1]$.*

If X, Y are both compact metric, then $f : X \rightarrow Y$ is κ -tight iff $\{y \in Y : |f^{-1}\{y\}| \geq \kappa\}$ is countable (see Theorem 2.7 of [12]). Of course, the \leftarrow direction is trivial. The \rightarrow direction for non-metric X and $\kappa = 2$ is refuted by the standard map from the double arrow space onto $[0, 1]$, which is tight by Lemma 2.3 of [12]:

Lemma 3.3 *If X, Y are compact LOTSEs and $f : X \rightarrow Y$ is order-preserving ($x_1 < x_2 \rightarrow f(x_1) \leq f(x_2)$), then f is tight.*

One can estimate the tightness of product maps using Lemma 2.14 of [12]:

Lemma 3.4 *Assume that for $i = 0, 1$: X_i, Y_i are compact, $f_i : X_i \rightarrow Y_i$ is $(m_i + 1)$ -tight, $m_i \leq n_i < \omega$, and $|f_i^{-1}\{y\}| \leq n_i$ for all $y \in Y_i$. Then $f_0 \times f_1 : X_0 \times X_1 \rightarrow Y_0 \times Y_1$ is $(\max(m_0 n_1, m_1 n_0) + 1)$ -tight.*

The notion of a *dissipated* compactum (Definition 3.11 below) involves tight maps onto metric compacta, ordered by *fineness*, so we define:

Definition 3.5 *Assume that X, Y, Z are compact, $f : X \rightarrow Y$, and $g : X \rightarrow Z$. Then $f \leq g$, or f is finer than g , iff there is a $\Gamma \in C(f(X), g(X))$ such that $g = \Gamma \circ f$.*

Lemma 3.6 *Assume that X, Y, Z are compact, $f : X \rightarrow Y$, and $g : X \rightarrow Z$. Then $f \leq g$ iff $\forall x_1, x_2 \in X [f(x_1) = f(x_2) \rightarrow g(x_1) = g(x_2)]$.*

Definition 3.7 *Assume X is compact. Let $\mathfrak{M}(X)$, the metric projections of X , be the class of all maps π such that $\pi : X \rightarrow Y$ for some compact metric Y . Then $\pi \in \mathfrak{M}_{\mathfrak{S}}(X) \subseteq \mathfrak{M}(X)$ iff in addition, each $\pi^{-1}\{y\}$ is scattered.*

Lemma 3.8 *If $\pi, \sigma \in \mathfrak{M}(X)$ and $\pi \leq \sigma \in \mathfrak{M}_\Theta(X)$, then $\pi \in \mathfrak{M}_\Theta(X)$.*

Observe that in the definition of $f \leq g$, it is irrelevant whether f, g map X onto Y, Z . Here, and in the definition of $\mathfrak{M}(X)$, we should really regard f in the set-theoretic sense as a set of ordered pairs, not as a triple (f, X, Y) , so that $f : X \rightarrow Y$ and $f : X \twoheadrightarrow f(X)$ are exactly the same object. One could also define $\mathfrak{M}(X)$ and $\mathfrak{M}_\Theta(X)$ as sets of closed equivalence relations on X .

Lemma 3.9 *$\mathfrak{M}(X)$ is countably directed. That is, if $\sigma_n \in \mathfrak{M}(X)$ for $n \in \omega$, then there is a $\pi \in \mathfrak{M}(X)$ with $\pi \leq \sigma_n$ for each n .*

Lemma 3.10 *If $\sigma \in \mathfrak{M}_\Theta(X)$, then there is a $\pi \in \mathfrak{M}_\Theta(X)$ with $\pi \leq \sigma$ and $\pi : X \twoheadrightarrow Y$, such that $\pi^{-1}\{b\}$ is a singleton for some $b \in Y$.*

Proof. Say $\sigma : X \twoheadrightarrow Z$. Fix any $c \in Z$, and then fix $a \in \sigma^{-1}\{c\}$ such that a is isolated in $\sigma^{-1}\{c\}$. Since Z is metric, $\{a\}$ is a G_δ in X , so fix any $f \in C(X, [0, 1])$ with $\{a\} = f^{-1}\{1\}$. Choose $\pi \in \mathfrak{M}_\Theta(X)$ with $\pi \leq \sigma$ and $\pi \leq f$. ☕

Only a scattered compactum X has the property that *all* maps in $\mathfrak{M}(X)$ are tight: If X is not scattered, then X maps onto $[0, 1]^2$; if we follow that map by the usual projection onto $[0, 1]$, we get a map from X onto $[0, 1]$ which is not even \mathfrak{c} -tight. The *dissipated* compacta have the property that *cofinally* many of these maps are tight:

Definition 3.11 *X is κ -dissipated iff X is compact and whenever $g \in \mathfrak{M}(X)$, there is a finer κ -tight $f \in \mathfrak{M}(X)$. X is dissipated iff X is 2-dissipated.*

So, the 1-dissipated compacta are the scattered compacta. Metric compacta are dissipated because we can let f be identity map. By Lemma 3.12 of [12]:

Lemma 3.12 *For any κ , the class of κ -dissipated compacta is a local class.*

An easy example of a dissipated space is given by:

Lemma 3.13 *If X is a compact LOTS, then X is dissipated*

The proof (see Lemma 3.4 of [12]) shows that given $g \in \mathfrak{M}(X)$, there is a finer $f \in \mathfrak{M}(X)$ such that $f(X)$ is a compact metric LOTS and f is order-preserving.

Note that just having *one* tight map g from X onto some metric compactum Z is not sufficient to prove that X is dissipated, since the tightness of g says nothing at all about the complexity of a particular $g^{-1}\{z\}$. However, if all $g^{-1}\{z\}$ are scattered, then just one tight g is enough by Lemma 3.5 of [12]:

Lemma 3.14 *Assume that some $g \in \mathfrak{M}_\Theta(X)$ is κ -tight. Then all $f \leq g$ are also κ -tight, so that X is κ -dissipated.*

This suggests the following definition:

Definition 3.15 $\pi \in \mathfrak{M}(X)$ is κ -supertight iff π is κ -tight and $\pi \in \mathfrak{M}_\Theta(X)$. Then X is κ -superdissipated iff some $\pi \in \mathfrak{M}_\Theta(X)$ is κ -supertight.

Using Lemmas 3.14, 3.12, and 3.8 above:

Lemma 3.16 *If $\pi, \sigma \in \mathfrak{M}_\Theta(X)$, $\pi \leq \sigma$, and σ is κ -supertight, then π is κ -supertight.*

Lemma 3.17 *A compactum X is κ -superdissipated iff X is κ -dissipated and $\mathfrak{M}_\Theta(X) \neq \emptyset$.*

Lemma 3.18 *The class of κ -superdissipated compacta is a local class.*

By Lemma 3.3:

Lemma 3.19 *The standard map $\sigma : I_\Lambda \rightarrow I$ is 2-supertight.*

The situation for products is more complicated. By Lemma 3.4 and induction:

Lemma 3.20 *For any $n \geq 1$ and $S_i \subseteq I$ (for $i < n$): The standard map $\sigma : \prod_{i < n} I_{S_i} \rightarrow I^n$ is $(2^{n-1} + 1)$ -supertight.*

This result is best possible by Theorem 3.9 of [12]; a product $\prod_{i < n} X_i$ is not (2^{n-1}) -dissipated if each X_i is a compact separable LOTS, none of the X_i is scattered, and at most one of the X_i is second countable.

Definition 3.21 *The perfect kernel, $\ker(X)$, is \emptyset if X is scattered, and the largest perfect subset of X otherwise.*

By Lemma 3.2, the tightness of $\pi : X \rightarrow Y$ can be expressed using perfect subsets of X , so that

Lemma 3.22 $\pi : X \rightarrow Y$ is κ -(super)tight iff $\pi \upharpoonright \ker(X)$ is κ -(super)tight, and the space X is κ -(super)dissipated iff $\ker(X)$ is κ -(super)dissipated.

Lemma 3.23 *Assume that $\pi : X \rightarrow Y$ is $(n + 2)$ -supertight, where $n \in \omega$, X is compact and Y is compact metric, and $\{P_0, \dots, P_n\}$ is a loose family for π of size $n + 1$, with each $\pi(P_j) = Q$. Then each $\pi \upharpoonright P_j : P_j \rightarrow Q$ is 2-supertight, $\ker(\pi^{-1}(Q)) \subseteq \bigcup_j P_j$, and $\pi^{-1}(Q)$ is 2-superdissipated.*

Proof. If tightness fails for $\pi \upharpoonright P_j$, then we could find uncountable closed $Q' \subseteq Q$ and disjoint closed $P_j^0, P_j^1 \subseteq P_j$ with $\pi(P_j^0) = \pi(P_j^1) = Q'$. If $P'_k = P_k \cap \pi^{-1}(Q')$, then the sets $P'_0, \dots, P'_{j-1}, P_j^0, P_j^1, P'_{j+1}, \dots, P'_n$ would be a loose family for π of size $n+2$. If $\ker(\pi^{-1}(Q)) \not\subseteq \bigcup_j P_j$, we could find a perfect $R \subseteq \pi^{-1}(Q) \setminus \bigcup_j P_j$; then $\pi(R)$ is non-scattered (since all $\pi^{-1}\{y\}$ are scattered), and R plus the P_j would contradict the $(n+2)$ -tightness of π . Finally, by Lemma 3.22, it is sufficient to prove that $\bigcup_j P_j$ is superdissipated, and this is done using the map into $Y \times \{0, 1, \dots, n\}$ which sends $x \in P_j$ to $(\pi(x), j)$. ☕

Finally, we mention two lemmas for the case that X does not contain a Cantor subset. $\pi : X \rightarrow Y$ is trivially n -supertight when all $|\pi^{-1}\{y\}| < n$, but also

Lemma 3.24 *Assume that $\pi : X \rightarrow Y$, X is compact, Y is compact metric, each $|\pi^{-1}\{y\}| \leq n$, and X has no Cantor subsets. Then π is n -supertight.*

Proof. If not, let $P_0, \dots, P_{n-1} \subseteq X$ be a loose family, with each $\pi(P_j) = Q$. Then Q has a Cantor subset, and each P_j is homeomorphic (via π) to Q . ☕

By the next lemma, the spaces X we consider are always totally disconnected:

Lemma 3.25 *Assume that $\pi : X \rightarrow Y$, where X is compact and Y is metric. Assume that each $\pi^{-1}\{y\}$ is totally disconnected and X does not contain a copy of the Cantor set. Then X is totally disconnected.*

Proof. Assume that X is not totally disconnected. Fix a metric on Y for which $\text{diam}(Y) \leq 1$. Obtain K_s for $s \in 2^{<\omega}$ to satisfy:

1. K_s is an infinite closed connected subset of X .
2. $\text{diam}(\pi(K_s)) \leq 2^{-\text{lh}(s)}$.
3. $K_{s \smallfrown 0}, K_{s \smallfrown 1} \subset K_s$ and $K_{s \smallfrown 0} \cap K_{s \smallfrown 1} = \emptyset$.

Assuming that this can be done, define $K_f = \bigcap_{n \in \omega} K_{f \upharpoonright n}$ for $f \in 2^\omega$. By (2), $|\pi(K_f)| = 1$; say $\pi(K_f) = \{y_f\}$. But K_f is connected and $\pi^{-1}\{y_f\}$ is totally disconnected, so $|K_f| = 1$; say $K_f = \{x_f\}$. Then $f \mapsto x_f$ is a homeomorphism from 2^ω into X , contradicting our assumptions about X .

To build the K_s : For $K_{\langle \rangle}$, just use the assumption that X is not totally disconnected. Now, say we are given K_s . Choose $x_0, x_1 \in K_s$ with $x_0 \neq x_1$. Then find disjoint relatively open $U_0, U_1 \subseteq K_s$ with each $x_\ell \in U_\ell$ and $\text{diam}(\pi(U_\ell)) \leq 2^{-\text{lh}(s)-1}$. Then find relatively open $V_\ell \subseteq K_s$ with $x_\ell \in V_\ell \subseteq \overline{V_\ell} \subseteq U_\ell$. Then, let $K_{s \smallfrown \ell}$ be the connected component of the point x_ℓ in the space $\overline{V_\ell}$, and note that $K_{s \smallfrown \ell}$ cannot be a singleton. ☕

4 The CSWP: Two Reductions

These reductions were described in [11]: Using the standard theory of function algebras (see [3, 4]), we can reduce the CSWP to the study of idempotents, and we can reduce the study of the CSWP in LOTSeS to the separable case.

If $f \in C(X)$, then f is an *idempotent* iff $f^2 = f$; equivalently iff f is the characteristic function of some clopen set. An idempotent is called *nontrivial* iff it is not the identically 0 or the identically 1 function. As with other proofs of the CSWP [5, 11], we shall proceed by considering idempotents. Following [11],

Definition 4.1 *The compact space X has the NTIP iff every $\mathcal{A} \sqsubseteq C(X)$ contains a non-trivial idempotent.*

So, the NTIP is trivially false of connected spaces. If X is not connected, then the CSWP implies the NTIP. The following is Lemma 3.5 of [11]; it is also easy to prove from the Bishop Antisymmetric Decomposition (see [2], or Theorem 13.1 in Chapter II of [3]).

Lemma 4.2 *Assume that X is compact and every perfect subset of X has the NTIP. Then X has the CSWP.*

Among the totally disconnected spaces, the NTIP is strictly weaker than the CSWP (see [11]). However, the lemma implies the following corollary, which is used to reduce proofs of the CSWP to proofs of the weaker NTIP:

Corollary 4.3 *If \mathfrak{K} is a closed-hereditary class of compact spaces and every perfect space in \mathfrak{K} has the NTIP, then every space in \mathfrak{K} has the CSWP.*

In particular, if \mathfrak{K} is the class of compact scattered spaces, then this corollary applies vacuously, so all spaces in \mathfrak{K} have the CSWP. If \mathfrak{K} contains some non-scattered spaces, then, as in [11, 5], we produce idempotents using:

Lemma 4.4 *Suppose that $\mathcal{A} \sqsubseteq C(X)$ and there is some $h \in \mathcal{A}$ such that either $\mathfrak{R}(h(X))$ or $\mathfrak{S}(h(X))$ is not connected. Then \mathcal{A} contains a non-trivial idempotent.*

This is easy to prove using Runge's Theorem; see Lemma 2.5 of [11], but the method was also used in [14] and [9].

It remains to describe how to obtain such an h . If X is scattered, then $\mathfrak{R}(h(X))$ is scattered also, so any h for which $\mathfrak{R}(h(X))$ is not a singleton will do; this is essentially the argument of [15]. In some other cases, we can obtain h using a tight map of X onto a metric space; this is described in Section 5.

We now turn to the second reduction. As in §5 of [11],

Definition 4.5 If μ is a regular complex Borel measure on the compact space X , then $|\mu|$ denotes its total variation, and $\text{supt}(\mu) = \text{supt}(|\mu|)$ denotes its (closed) support; that is, $\text{supt}(\mu) = X \setminus \bigcup\{U \subseteq X : U \text{ is open \& } |\mu|(U) = 0\}$.

Considering measure orthogonal to \mathcal{A} , we get:

Lemma 4.6 Assume that X is compact and that $\text{supt}(\mu)$ has the CSWP for all regular Borel measure μ . Then X has the CSWP.

By Corollary 5.4 of [12], every such $\text{supt}(\mu)$ is separable in the case that X is \aleph_0 -dissipated; for a LOTS X , this was a much earlier folklore result.

Corollary 4.7 If X fails the CSWP and is \aleph_0 -dissipated, then some compact separable subspace of X fails the CSWP.

Question 4.8 Is there a compact space X which fails the CSWP such that all compact separable subspaces of X satisfy the CSWP?

This X cannot be one of the three examples already known to fail the CSWP — namely, any space containing either the Cantor set [14] or $\beta\mathbb{N}$ [9] or the examples of [6, 7] (obtained assuming \diamond or CH), since all these spaces are separable.

Now, considering products of LOTSes:

Lemma 4.9 Assume that X is a compact subset of $\prod_{\alpha < \kappa} L_\alpha$, where each L_α is a LOTS, and assume that X does not have the CSWP. Then for some separable closed compact $H_\alpha \subseteq L_\alpha$, the space $X \cap \prod_{\alpha < \kappa} H_\alpha$ also fails the CSWP.

Proof. Let $\pi_\alpha : X \rightarrow L_\alpha$ be the usual coordinate projection. We may assume that each $L_\alpha = \pi_\alpha(X)$, so that L_α is compact. Fix μ on X such that $\text{supt}(\mu)$ fails the CSWP, let $\mu\pi_\alpha^{-1}$ be the induced measure on L_α , let $H_\alpha = \text{supt}(\mu\pi_\alpha^{-1})$, which is separable, and note that $\text{supt}(\mu) \subseteq X \cap \prod_{\alpha < \kappa} H_\alpha$. ☕

Lemma 4.10 For any $\kappa \leq \omega$: Suppose that there is a compact $X \subseteq \prod_{\alpha < \kappa} L_\alpha$, where each L_α is a LOTS, X has no Cantor subset, and X does not have the CSWP. Then there is such an X which is a subset of $(I_S)^\kappa$ for some $S \subseteq (0, 1)$.

Proof. By Lemma 4.9, we may assume that each L_α is separable and compact. Now, let L be the compact separable LOTS obtained by placing the L_α end-to-end, adding a point ∞ in the case that $\kappa = \omega$. Then we may assume that $X \subseteq L^\kappa$. Finally, replace L by an I_S using Lemma 2.5(3). ☕

5 The CSWP and Tightness

We show here how one can use the concepts from Section 3 to produce idempotents, and thus to prove the CSWP.

Definition 5.1 Assume that $\pi : X \rightarrow Y$, where X, Y are compact. Then, for $f \in C(X)$, define $\hat{f} = (\pi \times f)(X)$; that is,

$$\hat{f} = \{(\pi(x), f(x)) : x \in X\} \subseteq Y \times \mathbb{C}.$$

Lemma 5.2 Each \hat{f} is compact.

We plan to apply the next definition and lemma to sets of the form \hat{f} :

Definition 5.3 Fix $E \subseteq Y \times \mathbb{C}$ and $\Phi : \mathbb{C}^m \rightarrow \mathbb{C}$. Then $E_y = \{z : (y, z) \in E\}$ and

$$\Phi * E = \bigcup_{y \in Y} \Phi((E_y)^m) \subseteq \mathbb{C}.$$

Lemma 5.4 Suppose that $F \subseteq Y \times \mathbb{C}$ is compact and $\Phi : \mathbb{C}^m \rightarrow \mathbb{C}$ is continuous. Let \mathcal{B} be an open base for $Y \times \mathbb{C}$ which is closed under finite unions. Then $\Phi * F$ is compact and $\Phi * F = \bigcap \{\Phi * \bar{U} : U \in \mathcal{B} \text{ \& } F \subseteq U\}$.

Lemma 5.5 Assume that $\pi : X \rightarrow Y$ is n -supertight and $f \in C(X)$. Fix a continuous $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}$ such that $\Phi(z_1, \dots, z_n) = 0$ unless all n of the z_1, \dots, z_n are different. Then $\Phi * \hat{f}$ is compact and countable, and hence scattered.

Proof. Compactness follows from the compactness of X, Y . By n -tightness, $|\hat{f}_y| < n$, and hence $\Phi((\hat{f}_y)^n) = \{0\}$, for all but countably many y (see Lemma 3.2). But for all y , $\pi^{-1}\{y\}$ is scattered, so that $\Phi((\hat{f}_y)^n)$ is also scattered, and hence countable. Thus, the union of all these sets is also countable. ☕

Dissipation is a notion of smallness, which is balanced by a notion of bigness, which is really a partition property:

Definition 5.6 Fix a real $r > 0$. The compact space X is n -big iff for all $\mathcal{A} \subseteq C(X)$ and all partitions $\Upsilon : \mathcal{A} \rightarrow \omega$, there are $f_1, \dots, f_n \in \mathcal{A}$ and a point $c \in X$ such that the $\Upsilon(f_j)$, for $j = 1, \dots, n$, are all equal, and such that $|f_i(c) - f_j(c)| \geq r$ whenever $1 \leq i < j \leq n$.

Since \mathcal{A} is a linear subspace, it does not matter which $r > 0$ we use. The notion of 1-big is trivial, and 2-big is easily characterized:

Lemma 5.7 *The compact space X is 2-big iff X is not second countable.*

Proof. Note that $\exists c (|f_1(c) - f_2(c)| \geq r)$ holds iff $\|f_1 - f_2\| \geq r$. Also, if X is not second countable then $C(X)$ is not separable, and hence any $\mathcal{A} \subseteq C(X)$ is not separable, since the algebra generated by the functions in \mathcal{A} and their complex conjugates is dense in $C(X)$ by the Stone-Weierstrass Theorem. ☕

We relate this to the NTIP with the aid of:

Definition 5.8 *For each $n \geq 2$, define $\Xi_n : \mathbb{C}^n \rightarrow \mathbb{C}$ by:*

$$\Xi_n(z_1, \dots, z_n) = 2 \cdot \prod_{1 \leq i < j \leq n} (z_j - z_i) \quad .$$

Lemma 5.9 *Ξ_n is a polynomial in n variables. $\Xi(z_1, \dots, z_n) = 0$ unless all n of the z_1, \dots, z_n are different. If $|z_i - z_j| \geq 1$ for all $i < j \leq n$, then either $|\Re(\Xi_n(z_1, \dots, z_n))| \geq 1$ or $|\Im(\Xi_n(z_1, \dots, z_n))| \geq 1$.*

Lemma 5.10 *Assume that X is compact, $\mathcal{A} \subseteq C(X)$, $H \subseteq V \subseteq X$, where H, V are both clopen, and for some $n \geq 2$, V is n -superdissipated and H is n -big. Assume also that there is a $\psi \in \mathcal{A}$ such that $|\psi(x)| \leq 1/2$ for all $x \in X \setminus V$ and $|\psi(x)| \geq 1$ for all $x \in H$. Then \mathcal{A} has a non-trivial idempotent.*

Proof. Fix $\pi : V \twoheadrightarrow Y$ which is n -supertight. Applying Lemmas 3.10 and 3.16, we assume also that we have $b \in Y$ and $a \in V$ such that $\pi^{-1}\{b\} = \{a\}$. Let $R^+(z_1, \dots, z_n) = -R^-(z_1, \dots, z_n) = \Re(\Xi_n(z_1, \dots, z_n))$ and $I^+(z_1, \dots, z_n) = -I^-(z_1, \dots, z_n) = \Im(\Xi_n(z_1, \dots, z_n))$, so that $R^+, R^-, I^+, I^- : \mathbb{C}^n \rightarrow \mathbb{R}$. Call (E, ρ, τ) good iff:

1. $\rho, \tau \in \mathbb{Q}$ and $1/2 < \rho < \tau < 1$.
2. $E \subseteq Y \times \mathbb{C}$.
3. $[\rho, \tau]$ is disjoint from each of $R^+ * E, R^- * E, I^+ * E, I^- * E$.
4. $|\Xi_n(z_1, \dots, z_n)| < \rho$ whenever $z_1, \dots, z_n \in E_b$.

For $f \in C(X)$, use \widehat{f} for $\widehat{f|V}$. Observe that for each $f \in C(X)$, we may choose ρ, τ so that $(\widehat{f}, \rho, \tau)$ is good: (4) is no problem since \widehat{f}_b is a singleton. For the rest, note that each of $R^+ * \widehat{f}, R^- * \widehat{f}, I^+ * \widehat{f}, I^- * \widehat{f}$ is scattered by Lemma 5.5, so we may choose ρ, τ to make (1)(3) true.

Let \mathcal{B} be a countable open base for $Y \times \mathbb{C}$ which is closed under finite unions. For each $f \in \mathcal{A}$, choose $s = s_f \in \omega$ so that $|(\psi(x))^s f(x)| \leq 1/8$ for all $x \in X \setminus V$. Then, choose ρ_f, τ_f so that $(\widehat{\psi^{s_f} f}, \rho_f, \tau_f)$ is good. Then, applying Lemma 5.4, choose a $U_f \in \mathcal{B}$ such that (U_f, ρ_f, τ_f) is good and $\widehat{\psi^{s_f} f} \subseteq U_f$. Next,

apply the definition of “ n -big” using $\mathcal{A}|H$: Fix $c \in H$ and $f_1, f_2, \dots, f_n \in \mathcal{A}$ and (U, ρ, τ, s) such that $(U_{f_j}, \rho_{f_j}, \tau_{f_j}) = (U, \rho, \tau)$ and $s_{f_j} = s$ for all j , and also $|f_j(c) - f_k(c)| \geq 1$, and hence $|\psi^s(c)f_j(c) - \psi^s(c)f_k(c)| \geq 1$, whenever $j \neq k$.

Let $h(x) = \Xi_n((\psi(x))^s f_1(x), \dots, (\psi(x))^s f_n(x))$; then $h \in \mathcal{A}$. Then, choose $\Phi \in \{\mathbb{R}^+, \mathbb{R}^-, \mathbb{I}^+, \mathbb{I}^-\}$ so that $\Phi((\psi(c))^s f_1(c), \dots, (\psi(c))^s f_n(c)) \geq 1$, and let $k(x) = \Phi((\psi(x))^s f_1(x), \dots, (\psi(x))^s f_n(x))$; so $k(x)$ is either $\pm \Re(h(x))$ or $\pm \Im(h(x))$.

Note that when $x \in X \setminus V$, each $|(\psi(x))^s f_j(x) - (\psi(x))^s f_k(x)| \leq 1/4$ so (referring to the definition of Ξ), $|h(x)| \leq 1/2$. Then $k(X) = k(V) \cup k(X \setminus V) \subseteq \Phi * U \cup [-1/2, 1/2]$ is disjoint from $[\rho, \tau]$, but contains $k(c) > \tau$ and $k(a) < \rho$. Thus, either $\Re(h(X))$ or $\Im(h(X))$ is not connected, so \mathcal{A} contains a non-trivial idempotent by Lemma 4.4. ☕

In this section, we use only the special case of this lemma where $H = V = X$, in which case the hypotheses on ψ are trivial, and the above proof can be simplified somewhat. The more general result will be needed in Section 6.

Setting $H = V = X$, we have:

Lemma 5.11 *Suppose that $n \geq 2$ and X is both n -big and n -superdissipated. Then X has the NTIP.*

Applying this and Lemma 5.7, we have:

Theorem 5.12 *If X is 2-superdissipated and is not second countable, then X has the NTIP.*

This theorem yields the NTIP for some spaces not covered by [5, 11], but the result on CSWP, obtained from Corollary 4.3, is contained in the results of [5]:

Corollary 5.13 *If X is 2-superdissipated and does not contain a Cantor subset, then X has the CSWP.*

The examples of [6, 7] show (under \diamond or CH) that this need not hold if X is merely 2-dissipated. To extend this corollary to 3-superdissipated spaces, we need a mechanism (Lemma 5.15) for proving that a space is 3-big. This notion, unlike 2-big (see Lemma 5.7), does not seem to have a simple equivalent in terms of standard cardinal functions; see Section 8.

Lemma 5.14 *Assume that $n \geq 1$ and that X is $(n+2)$ -superdissipated but not $(n+1)$ -superdissipated, and then fix $\sigma : X \twoheadrightarrow Z$ which is $(n+2)$ -supertight, where Z is compact metric. Assume that X does not have a Cantor subset. Fix $\mathcal{A} \subseteq C(X)$ and $\Upsilon : \mathcal{A} \rightarrow \omega$. Fix any disjoint open sets $V_0, V_1, V_2 \subseteq \mathbb{C}$ and any $\pi \in \mathfrak{M}(X)$. Then there are f, g, a, d, c such that:*

1. $f, g \in \mathcal{A}$ and $\Upsilon(f) = \Upsilon(g)$.
2. $a, d \in X$, $c = \sigma(a) = \sigma(d) \in Z$, and $\pi(a) = \pi(d)$.
3. $f(a) \in V_0$ and $g(a) \in V_1$.
4. $f(d) \in V_2$ and $g(d) \in V_2$.
5. For all $x \in \sigma^{-1}\{c\}$, $(f(x), g(x)) \in V_0 \times V_1 \cup V_2 \times V_2$.

Proof. First, replacing π by a finer map, we may assume that $\pi \leq \sigma$, so that $\pi \in \mathfrak{M}_\mathfrak{S}(X)$ and π also is $(n+2)$ -supertight (see Lemmas 3.8, 3.9, and 3.16). Say $\pi : X \rightarrow Y$; then fix $\Gamma \in C(Y, Z)$ with $\sigma = \Gamma \circ \pi$.

Since π is not $(n+1)$ -supertight, fix a loose family for π , $\{P_0, \dots, P_n\}$, with each $\pi(P_j) = Q$ and each P_j perfect (see Lemma 3.2). Then $\{P_0, \dots, P_n\}$ is also a loose family for σ , with each $\sigma(P_j) = \Gamma(Q)$; note that $\Gamma(Q)$ cannot be scattered since Q is not scattered and each $\Gamma^{-1}\{z\}$ is scattered. Then $\sigma^{-1}(\Gamma(Q)) = \pi^{-1}(\Gamma^{-1}(\Gamma(Q)))$ is superdissipated by Lemma 3.23, so it has the CSWP by Corollary 5.13. Also, X is totally disconnected by Lemma 3.25. Fix closed disjoint $\tilde{P}_j \subseteq \sigma^{-1}(\Gamma(Q))$ such that each $\tilde{P}_j \supseteq P_j$ and $\bigcup_j \tilde{P}_j = \sigma^{-1}(\Gamma(Q))$. Note that each $\sigma \upharpoonright \tilde{P}_j$ is supertight by Lemma 3.23.

Choose $y_\xi \in Q$ for $\xi < \omega_1$ such that the $\Gamma(y_\xi)$ are all different and each $|\pi^{-1}\{y_\xi\} \cap P_0| \geq 2$; this is possible because P_0 does not have a Cantor subset. Then, applying the CSWP for $\sigma^{-1}(\Gamma(Q))$, choose $h_\xi \in \mathcal{A}$ for $\xi < \omega_1$ such that $h_\xi(\tilde{P}_0) \subseteq V_0 \cup V_1$, $h_\xi(\tilde{P}_j) \subseteq V_2$ when $j \geq 1$, and $h_\xi(\pi^{-1}\{y_\xi\} \cap P_0)$ meets both V_0 and V_1 . Since there are only countably many values for Υ , we may assume that the $\Upsilon(h_\xi)$ are all the same. For each ξ , we have P_0 partitioned into two relatively clopen sets, $h_\xi^{-1}(V_0) \cap P_0$ and $h_\xi^{-1}(V_1) \cap P_0$, and both these sets meet $\pi^{-1}\{y_\xi\}$. If these clopen partition were the same for all ξ , we would contradict the tightness of $\pi \upharpoonright P_0$ (see Lemma 3.23), so that we may fix $\xi \neq \eta$ with $H := h_\xi^{-1}(V_0) \cap h_\eta^{-1}(V_1) \cap P_0$ non-empty, and thus perfect. Let $f = h_\xi$ and $g = h_\eta$.

If we choose any $a \in H$, we may set $c = \sigma(a)$, and choose any $d \in P_1 \cap \pi^{-1}\{\pi(a)\}$. This will satisfy (1)(2)(3)(4), but (5) might fail, since there may be an $x_c \in \sigma^{-1}\{c\}$ such that $x_c \in \tilde{P}_0$ and either $f(x_c) \in V_1$ or $g(x_c) \in V_0$. But note that we also have $a \in \sigma^{-1}\{c\}$ and $a \in \tilde{P}_0$ and $f(a) \in V_0$ and $g(a) \in V_1$. Consider the map $(f, g) : X \rightarrow \mathbb{C} \times \mathbb{C}$. If (5) fails for every choice of $a \in H$, then there would be uncountably many $c \in \pi(H)$ such that (f, g) takes more than one value on $\tilde{P}_0 \cap \sigma^{-1}\{c\}$, contradicting the tightness of $\sigma \upharpoonright \tilde{P}_0$. Thus, we may choose a, b, d so that (1)(2)(3)(4)(5) hold. ☕

Lemma 5.15 *Assume that X is not dissipated, but that X is m -superdissipated for some $m \in \omega$, and that X does not have a Cantor subset. Then X is 3-big.*

Proof. Fix $\mathcal{A} \subseteq C(X)$ and $\Upsilon : \mathcal{A} \rightarrow \omega$. Fix any disjoint open sets $V_0, V_1, V_2 \subseteq \mathbb{C}$. To verify that X is 3-big, it is sufficient to find $h_0, h_1, h_2 \in \mathcal{A}$ and $x \in X$ such that each $h_j(x) \in V_j$.

Fix $n \geq 1$ such that X is $(n+2)$ -superdissipated but not $(n+1)$ -superdissipated, and then fix $\sigma : X \twoheadrightarrow Z$ which is $(n+2)$ -supertight. Let \mathcal{B} be a countable open base for Z . For $\pi \in \mathfrak{M}(X)$, call $F = (f, g, a, d, c, s, U) = (f^F, g^F, a^F, d^F, c^F, s^F, U^F)$ good for π iff (1–5) from Lemma 5.14 hold together with:

6. $s \in \omega$ and $\Upsilon(f) = \Upsilon(g) = s$.
7. $c \in U$, $U \in \mathcal{B}$, and for all $x \in \sigma^{-1}(\overline{U})$, $(f(x), g(x)) \in V_0 \times V_1 \cup V_2 \times V_2$.

Such an F always exists. To see this, first get (f, g, a, d, c) by Lemma 5.14 to satisfy (1–5). Then (6) is trivial, and we choose U to satisfy (7) using the fact that $\{z \in Z : \forall x \in \sigma^{-1}\{z\} [(f(x), g(x)) \in V_0 \times V_1 \cup V_2 \times V_2]\}$ is open.

Note that if F is good for π and $\pi \leq \varphi$ then F is good for φ .

Next, note that there are fixed s and U such that for all $\pi \in \mathfrak{M}(X)$, there is an F good for π with $s^F = s$ and $U^F = U$: If not, then for each s, U , choose $\varphi_{s,U}$ such that no F good for φ satisfies $s^F = s$ and $U^F = U$. Then fix π such that $\pi \leq \varphi_{s,U}$ for each s, U . An F which is good for π yields a contradiction.

For each π , choose F^π good for π with $s^{F^\pi} = s$ and $U^{F^\pi} = U$, and write $(f^\pi, g^\pi, a^\pi, d^\pi, c^\pi)$ for $(f^{F^\pi}, g^{F^\pi}, a^{F^\pi}, d^{F^\pi}, c^{F^\pi})$.

Now, for each π , we have $\sigma^{-1}(\overline{U})$ partitioned into two relatively clopen sets, $A^\pi = \{x \in \sigma^{-1}(\overline{U}) : (f^\pi(x), g^\pi(x)) \in V_0 \times V_1\}$ and $D^\pi = \{x \in \sigma^{-1}(\overline{U}) : (f^\pi(x), g^\pi(x)) \in V_2 \times V_2\}$. If these are all the same, say $A^\pi = A$ and $D^\pi = D$ for all π ; then we may fix $\pi \in C(X, [0, 1])$ which is 0 on A and 1 on D , so $\pi(a^\pi) = 0$ and $\pi(d^\pi) = 1$, contradicting (2). Thus, we can choose π, φ and an $x \in A^\pi \cap D^\varphi$; then $f^\pi(x) \in V_0$, $g^\pi(x) \in V_1$, $f^\varphi(x) \in V_2$, as required. ☕

The “obvious” generalization of this would say that if X does not have a Cantor subset and is $(n+2)$ -superdissipated but not $(n+1)$ -superdissipated, then X is $(n+2)$ -big. For $n = 1$ this is Lemma 5.15, and for $n = 0$ this is Lemma 5.7. Unfortunately, this is not true in general; see Example 8.5. We do get:

Theorem 5.16 *Assume that X is compact and is 3-superdissipated and does not have a Cantor subset. Then X has the CSWP.*

Proof. Since “3-superdissipated” is closed-hereditary, it is sufficient, by Corollary 4.3, to assume that X is also perfect and prove that X has the NTIP. X cannot be second countable, so X is 2-big by Lemma 5.7. If X is not 2-superdissipated, then X is 3-big by Lemma 5.15. Thus, whether or not X is 2-superdissipated, it has the NTIP by Lemma 5.11. ☕

Corollary 5.17 *If X is compact and $X \subseteq L_0 \times L_1$, where L_0, L_1 are a LOTSeS, then X has the CSWP iff X does not contain a copy of the Cantor set.*

Proof. By Lemma 4.10, we may assume that $X \subseteq (I_S)^2$. Then X is 3-superdissipated by Lemma 3.20, so X has the CSWP by Theorem 5.16. ☕

We now can extend this to products of three LOTSeS, using an argument which is much more specific to ordered spaces. First, we introduce a notation for lines, boxes, etc. in such products.

Definition 5.18 *Let $\prod_{\alpha < \kappa} L_\alpha$ be a product of LOTSeS, and use $<$ for the order on each L_α . Then:*

- ☞ *If $\beta < \kappa$ and c is a point in $\prod_{\alpha \neq \beta} L_\alpha$, then $\text{line}(\beta, c) = \{x \in \prod_{\alpha < \kappa} L_\alpha : \forall \alpha \neq \beta [x_\alpha = c_\alpha]\}$. A line in $\prod_{\alpha < \kappa} L_\alpha$ is any set of the form $\text{line}(\beta, c)$.*
- ☞ *$<^+$ is $<$; $<^-$ is $>$; \leq^+ is \leq ; \leq^- is \geq .*
- ☞ *$\mathcal{D} = \{+, -\}^\kappa$ is the set of all directions. For $\Delta \in \mathcal{D}$ and $x, y \in \prod_{\alpha < \kappa} L_\alpha$, $x <^\Delta y$ iff $\forall \alpha [x_\alpha <^{\Delta_\alpha} y_\alpha]$ and $x \leq^\Delta y$ iff $\forall \alpha [x_\alpha \leq^{\Delta_\alpha} y_\alpha]$.*
- ☞ *If $a, b \in \prod_{\alpha < \kappa} L_\alpha$, then $\text{box}[a, b] = \prod_{\alpha < \kappa} [\min(a_\alpha, b_\alpha), \max(a_\alpha, b_\alpha)]$, and a (closed) box is any set of this form.*
- ☞ *If $a \in \prod_{\alpha < \kappa} L_\alpha$ and $\Delta \in \mathcal{D}$, then $\text{corn}(a, \Delta) = \{x \in \prod_{\alpha < \kappa} L_\alpha : a \leq^\Delta x\}$.*
- ☞ *If $a \in B \subseteq \prod_{\alpha < \kappa} L_\alpha$ and $\Delta \in \mathcal{D}$, then $\text{corn}(a, B, \Delta) = B \cap \text{corn}(a, \Delta)$.*

For example, in \mathbb{R}^3 : $(2, 4, 6) <^{+-+} (3, 3, 7) \leq^{+-+} (4, 2, 7)$. Now, let $B = [0, 9]^3 = \text{box}[(0, 0, 0), (9, 9, 9)] = \text{box}[(9, 0, 9), (0, 9, 0)]$. Then $\text{corn}((2, 4, 6), B, +++)$ is the box $[2, 9] \times [0, 4] \times [6, 9]$. The directions $\Delta \in \mathcal{D}$ are also useful inside products of the form $(I_S)^\kappa$. Continuing the notation of Definition 2.1,

Definition 5.19 *If $\sigma : (I_S)^\kappa \rightarrow I^\kappa$ is the standard map, $y \in I^\kappa$, and $\Delta \in \mathcal{D}$, then $y^\Delta = \langle y^{\Delta_\alpha} : \alpha < \kappa \rangle$.*

For example, if $b = (b_0, b_1, b_2) \in I^3$, then $\sigma^{-1}\{b\}$ consists of the points, $b^{\pm\pm\pm} = (b_0^\pm, b_1^\pm, b_2^\pm)$; e.g., b^{+-+} denotes the point $(b_0^+, b_1^-, b_2^+) \in (I_S)^3$. The size of $\sigma^{-1}\{b\}$ will be 8, 4, 2 or 1 depending on whether 3, 2, 1 or 0 of the b_0, b_1, b_2 lie in S .

The following lets us establish bigness for subsets of $(I_S)^n$ by checking a simpler geometric property:

Lemma 5.20 *Fix $S \subseteq (0, 1)$, a closed $X \subseteq (I_S)^n$, and m with $2^{n-1} < m \leq 2^n$. Assume that whenever $\Upsilon : S^n \rightarrow \omega$, there are distinct $\Delta_1, \Delta_2, \dots, \Delta_m \in \mathcal{D}$, a point $x \in X$, and $d_1, d_2, \dots, d_m \in S^n$ such that $x \in \text{corn}(d_j^{\Delta_j}, \Delta_j)$ for each j , and such that $\Upsilon(d_1) = \Upsilon(d_2) = \dots = \Upsilon(d_m)$. Then X is m -big.*

Proof. Note that for $d \in S^n$, the points $d^\Delta \in (I_S)^n$, for $\Delta \in \mathcal{D}$, are all distinct, and the $\text{corn}(d^\Delta, \Delta)$, for $\Delta \in \mathcal{D}$, partition $(I_S)^n$ into 2^n clopen subsets.

Fix $\mathcal{A} \subseteq C(X)$ and $\Upsilon : \mathcal{A} \rightarrow \omega$. Since finite spaces have the CSWP, we may choose, for each $d \in S^n$, an $f_d \in C((I_S)^n)$ with $f_d \upharpoonright X \in \mathcal{A}$ such that the $f_d(d^\Delta)$, for $\Delta \in \mathcal{D}$, are 2^n distinct integers. We shall verify the definition of “ m -big” just by considering the functions $f_d \upharpoonright X$; the r in Definition 5.6 will be $1/2$.

Each f_d is continuous, so choose $p(d), q(d) \in \mathbb{Q}^n$ with $\forall \mu [p(d)_\mu < d_\mu < q(d)_\mu]$ such that $\sup\{|f_d(x) - f_d(d^\Delta)| : x \in \text{corn}(d^\Delta, \text{box}[p(d)^+, q(d)^-], \Delta)\} \leq 1/4$ for each $\Delta \in \mathcal{D}$. Here, for $y \in I^n$, y^+ abbreviates $(y_0^+, \dots, y_{n-1}^+)$ and y^- abbreviates $(y_0^-, \dots, y_{n-1}^-)$. Now, let $\Upsilon'(d) = (\Upsilon(f_d \upharpoonright X), p(d), q(d))$. Since $\text{ran}(\Upsilon')$ is countable, we may apply the hypotheses of the lemma and fix distinct $\Delta_1, \Delta_2, \dots, \Delta_m \in \mathcal{D}$, along with $x \in X$ and $d_1, d_2, \dots, d_m \in S^n$, such that $x \in \text{corn}(d_j^{\Delta_j}, \Delta_j)$ for each j , $\Upsilon(f_{d_1} \upharpoonright X) = \Upsilon(f_{d_2} \upharpoonright X) = \dots = \Upsilon(f_{d_m} \upharpoonright X)$, and also each $p(d_j) = p$ and $q(d_j) = q$ for some $p, q \in \mathbb{Q}^n$.

Note that all $d_j^{\Delta_j} \in \text{box}[p^+, q^-]$. Also, since $m > 2^{n-1}$, $\{\Delta_1, \Delta_2, \dots, \Delta_m\}$ contains both Δ and $-\Delta$ for some Δ , which implies (using $x \in \text{corn}(d_j^{\Delta_j}, \Delta_j)$) that $x \in \text{box}[p^+, q^-]$. Thus, $x \in \text{corn}(d_j^{\Delta_j}, \text{box}[p^+, q^-], \Delta_j)$, so $|f_{d_j}(x) - f_{d_j}(d_j^{\Delta_j})| \leq 1/4$ for each j , so that $|f_{d_j}(x) - f_{d_k}(x)| \geq 1/2$ when $j \neq k$. ☕

Note that the points $d_j^{\Delta_j}$ were not assumed to lie in X .

Lemma 5.21 *Assume that $S \subseteq (0, 1)$, X is a closed subspace of $(I_S)^3$, X is not 3-dissipated, and X does not contain a Cantor subset. Then X is 6-big.*

Proof. We verify the hypotheses of Lemma 5.20, so fix $\Upsilon : S^3 \rightarrow \omega$; we must find appropriate $\Delta_1, \Delta_2, \dots, \Delta_6 \in \mathcal{D} = \{+, -\}^3$, $x \in X$, and $d_1, d_2, \dots, d_6 \in S^3$.

Note that it is sufficient to find x along with points $c_E, c_F, c_G \in S \times S$, numbers $u_E, w_E, u_G, w_G, u_F, v_F, w_F \in S$, and $\Delta \in \{+, -\}^2$ such that:

1. $c_E <^\Delta c_F <^\Delta c_G$.
2. $u_E, u_F, u_G < v_F$ and $v_F < w_E, w_F, w_G$.
3. Υ has the same value on the 6 points: $d_1 = (c_E, u_E)$, $d_2 = (c_E, w_E)$, $d_3 = (c_F, u_F)$, $d_4 = (c_F, w_F)$, $d_5 = (c_G, u_G)$, $d_6 = (c_G, w_G)$.
4. $x \in X$ and x is one of the four points (c_F^Γ, v_F^\pm) , where $\Gamma \in \{+, -\}^2$ and Γ is different from Δ and $-\Delta$.

Note that no ordering is assumed among u_E, u_F, u_G or among w_E, w_F, w_G . To verify that (1–4) are sufficient, and to clarify our notation, assume WLOG that $\Delta = ++$, so $c_E <^{++} c_F <^{++} c_G$. Then Γ is either $+-$ or $-+$; WLOG $\Gamma = +-$, so we are assuming X contains at least one of the two points (c_F^{+-}, v_F^\pm) , denoted by x . But now we obtain the hypotheses of Lemma 5.20. Namely, $x \in \text{corn}(d_j^{\Delta_j}, \Delta_j)$

for $j = 1, 2, \dots, 6$, setting $\Delta_1 = +++$, $\Delta_2 = ++-$, $\Delta_3 = +-+$, $\Delta_4 = +--$, $\Delta_5 = --+$, $\Delta_6 = ---$

Now, to obtain (1-4): If $E \subseteq S$, let $\sigma_E : (I_S)^3 \rightarrow (I_E)^3$ be the natural map; so $\sigma_\emptyset = \sigma$. If also $E \in [S]^\omega$ (i.e., $|E| = \aleph_0$), then $(I_E)^3$ is a compact metric space, and we shall use the fact that *none* of these σ_E are 3-tight.

If $E_1 \subseteq E_2 \in [S]^\omega$ then $\sigma_{E_2} \leq \sigma_{E_1}$ (see Lemma 3.6). Observe that $[S]^\omega$ is countably directed upward. Call $\mathcal{U} \subseteq [S]^\omega$ *cofinal* iff $\forall E_1 \in [S]^\omega \exists E_2 \in \mathcal{U} (E_1 \subseteq E_2)$; then \mathcal{U} is also countably directed upward. We shall use this observation several times to show that a number of quantities dependent on E can in fact be chosen uniformly, independently of E , on a cofinal set.

Temporarily fix an $E \in [S]^\omega$. Then we have $P_j = P_j^E \subseteq X \subseteq (I_S)^3$ for $j = 0, 1, 2$ such that $\{P_0, P_1, P_2\}$ is a loose family. Then each $\sigma_E(P_j) = Q$, where $Q = Q^E \subseteq \sigma_E(X) \subseteq (I_E)^3$ is uncountable. We can now get such a Q to be of a very simple form:

First, note that Q must be a subset of finite union of lines. If not, then we may choose $y^\ell = (y_0^\ell, y_1^\ell, y_2^\ell) \in Q$ for $\ell \in \omega$ such that no two of the y^ℓ lie on the same line; that is, whenever $\ell < m < \omega$, the triples y^ℓ and y^m differ on at least two coordinates. Now, we may thin the sequence and permute the coordinates and assume that each of the two sequences $\langle y_0^\ell : \ell \in \omega \rangle$ and $\langle y_1^\ell : \ell \in \omega \rangle$ is either strictly increasing or strictly decreasing, while $\langle y_2^\ell : \ell \in \omega \rangle$ is either constant or strictly increasing or strictly decreasing. If H is the set of limit points of the sequence of sets $\langle \sigma^{-1}\{y_i^\ell\} : \ell \in \omega \rangle$, then $|H| \leq 2$, but H must meet each of P_0, P_1, P_2 , which is a contradiction.

Next, shrinking Q , along with P_0, P_1, P_2 , we may assume that $Q = Q^E$ is a subset of one line; say $Q^E \subseteq \text{line}(\beta_E, c_E)$, where $\beta_E < 3$.

β_E depends on E , but since $[S]^\omega$ is countably directed upward, there is a fixed β such that $\beta_E = \beta$ on a cofinal set $\mathcal{U} \subseteq [S]^\omega$. By permuting coordinates, we may assume $\beta = 2$, so that $Q^E \subseteq \text{line}(2, c_E) \subseteq (I_E)^3$, where $c_E = (a_E, b_E) \in (I_E)^2$. From now on, we shall delete the “2”; so $\text{line}(c_E) = \{(a_E, b_E, u) : u \in I_E\}$. Then $Q^E = \{c_E\} \times \tilde{Q}_E$, where $\tilde{Q}_E \subseteq I_E$.

Again, fix E , and temporarily delete some of the sub/super-script E . Now $\sigma_E^{-1}(\text{line}(c)) \subseteq (I_S)^3$ is a union of 1, 2, or 4 lines in $(I_S)^3$. However, the existence of Q, P_0, P_1, P_2 implies that $\sigma_E : \sigma_E^{-1}(\text{line}(c)) \rightarrow \text{line}(c)$ is not 3-tight, so in fact $\sigma_E^{-1}(\text{line}(c))$ is a union of 4 lines, which means that $a, b \in S \setminus E$; that is, we may regard a, b as real numbers which are not split in I_E , but which are split into a^\pm, b^\pm in I_S , and $\sigma_E^{-1}(Q) \subseteq \text{line}(c^{++}) \cup \text{line}(c^{+-}) \cup \text{line}(c^{-+}) \cup \text{line}(c^{--}) \subseteq (I_S)^3$. Now $\sigma_E^{-1}(Q) \cap \text{line}(c^{++}) \cap X$ is some closed subset of $\sigma_E^{-1}(Q) \cap \text{line}(c^{++})$, but replacing Q by a smaller perfect set, we may assume that this closed subset is either empty or all of $\sigma_E^{-1}(Q) \cap \text{line}(c^{++})$. Repeating this argument three more times, we may assume that each of the four sets $\sigma_E^{-1}(Q) \cap \text{line}(c^{\pm\pm})$ is either

contained in X or disjoint from X . Again, the existence of P_0, P_1, P_2 implies that $\sigma_E : \sigma_E^{-1}(Q) \cap X \rightarrow Q$ is not 3-tight, so at least three of the four sets $\sigma_E^{-1}(Q) \cap \text{line}(c^{\pm\pm})$ are contained in X . Which three or four depends on E ; there is a cofinal set on which it is the same, although this is irrelevant now. More importantly, since $\tilde{Q}_E \subseteq I_E$ and E is countable, we may shrink Q^E and assume that $\tilde{Q}_E \cap E = \emptyset$; that is, we may regard \tilde{Q}_E as a perfect subset of $I \setminus E$. Note that S must meet every perfect subset of \tilde{Q} , since otherwise X would contain a Cantor subset. In particular, $S \cap \tilde{Q}$ is uncountable. Now $c_E = c = (a, b)$ is fixed, and for each $u \in S \cap \tilde{Q}$, we have the triple $d = d_u = (a, b, u)$. We may now choose $t \in \omega$ and $u, v, w \in S \cap \tilde{Q}$ such that $u < v < w$ and $\Upsilon(d_u) = \Upsilon(d_v) = \Upsilon(d_w) = t$. Also choose rational ρ, τ with $u < \rho < v < \tau < w$.

Of course, t, ρ, τ, u, v, w depend on E , but there are only \aleph_0 possibilities for t, ρ, τ , so we may assume that for E in our cofinal set \mathcal{U} , these are always the same, whereas u, v, w are really u_E, v_E, w_E .

Choose an increasing ω_1 sequence $\langle E_\xi : \xi < \omega_1 \rangle$ of elements of \mathcal{U} such that $\xi < \eta \rightarrow c_{E_\xi} \in (E_\eta)^2$. Now $c_{E_\xi} = (a_{E_\xi}, b_{E_\xi})$ and $a_{E_\xi}, b_{E_\xi} \notin E_\xi$, so $a_{E_\eta} \neq a_{E_\xi}$ and $b_{E_\eta} \neq b_{E_\xi}$ whenever $\xi \neq \eta$. It follows that we may find distinct $\xi_n < \omega_1$ for $n \in \omega$ and a fixed $\Delta \in \{+, -\}^2$ such that $m < n \rightarrow c_{E_{\xi_m}} <^\Delta c_{E_{\xi_n}}$. But, we only need three of these, so let E, F, G denote $E_{\xi_0}, E_{\xi_1}, E_{\xi_2}$. Then we have $c_E <^\Delta c_F <^\Delta c_G$ as in (1) above. $u_E, u_F, u_G < \rho < v_F < \tau < w_E, w_F, w_G$, so (2) holds. (3) holds because Υ has the same value t on all $(a_{E_\xi}, b_{E_\xi}, u_{E_\xi}), (a_{E_\xi}, b_{E_\xi}, v_{E_\xi}), (a_{E_\xi}, b_{E_\xi}, w_{E_\xi})$. Finally, we may choose x to make (4) hold because at least three of the four sets $\sigma_F^{-1}(\tilde{Q}_F) \cap \text{line}(c_F^\Gamma)$ (for $\Gamma \in \{+, -\}^2$) are contained in X and $v_F \in S \cap \tilde{Q}_F$, and for these Γ , both points (c_F^Γ, v_F^\pm) lie in X . ☕

Proof of Theorem 1.5. By Lemma 4.10, we may assume that $X \subseteq (I_S)^3$. Since the properties assumed of X are closed-hereditary, it is sufficient, by Corollary 4.3, to assume that X is also perfect and prove that X has the NTIP. Note that “dissipated” is the same as “superdissipated” for these spaces. If X is 3-dissipated, then X has the CSWP, and hence the NTIP, by Theorem 5.16. If X is not 3-dissipated, then X is 5-big by Lemma 5.21, but it also is 5-dissipated by Lemma 3.20, so X has the NTIP by Lemma 5.11. ☕

We do not know if the same theorem holds when X is contained in a product of four LOTSeS, but the analogue of Lemma 5.21 is false. That is, there is (see Example 8.6) a closed $X \subseteq (I_S)^4$ such that X is not 8-dissipated and is not 7-big. Of course, X must be 9-dissipated, but to prove the NTIP by our methods, X would need to be 9-big.

6 Removable Spaces

The property of a compact space being *removable*, defined in [5], is a strengthening of the CSWP. Many of the spaces proved in Section 5 to have the CSWP are in fact removable. We recall the definition, which is in terms of the Šilov boundary:

Definition 6.1 *If $\mathcal{A} \subseteq C(X)$, then $\text{III}(\mathcal{A})$ denotes the Šilov boundary; this is the smallest non-empty closed $H \subseteq X$ such that $\|f\| = \sup\{|f(x)| : x \in H\}$ for all $f \in \mathcal{A}$.*

This is discussed in texts on function algebras; see [3, 4]. Note that $\text{III}(\mathcal{A})$ cannot be finite unless X is finite, in which case $\text{III}(\mathcal{A}) = X$.

Definition 6.2 *A compact space K is removable iff for all X, U, \mathcal{A} , if:*

- ☞ X is compact, $U \subsetneq X$, and U is open,
- ☞ \overline{U} is homeomorphic to a subspace of K , and
- ☞ $\mathcal{A} \subseteq C(X)$ and all idempotents of \mathcal{A} are trivial,

then $\text{III}(\mathcal{A}) \subseteq X \setminus U$.

The next four lemmas are clear from [5]:

Lemma 6.3 *If X is removable, then X is totally disconnected and has the CSWP.*

It is unknown whether the converse to this lemma is true. The removable spaces are of interest because one can prove some theorems about them which are currently unknown for the CSWP spaces. In particular, the removable spaces form a local class (see Definition 1.7); this follows from:

Lemma 6.4 *If the compact X is a finite union of closed sets, each of which is removable, then X is removable.*

More generally, one can do a type of Cantor-Bendixson analysis for a compact X , iteratively deleting open sets with removable closures; if one gets to \emptyset , then X itself is removable and hence has the CSWP (see [5], Lemma 2.15). This results in the next definition and lemma.

Definition 6.5 *A compact space P is nowhere removable iff \overline{W} is not removable for all non-empty open $W \subseteq P$.*

Lemma 6.6 *If X is compact and not removable, then there is a non-empty closed $P \subseteq X$ such that P is nowhere removable.*

In particular, since the one-point space is removable,

Lemma 6.7 *Every compact scattered space is removable.*

Definition 6.8 \mathfrak{R} is the class of all compact spaces X such that for all perfect $H \subseteq X$: There is non-empty relatively clopen $U \subseteq H$ such that either U is removable or for some finite $n \geq 2$, U is both n -big and n -superdissipated.

If X is removable, then $X \in \mathfrak{R}$, and we shall soon prove the converse statement. No space in \mathfrak{R} can contain a Cantor subset (since the Cantor set is neither 2-big nor removable). All spaces in \mathfrak{R} are totally disconnected by Lemma 3.25.

Our proof will use the following restatement of Definition 6.2:

Lemma 6.9 *Assume that \mathfrak{K} is a closed-hereditary class of totally disconnected compact spaces, and assume that whenever Z, V, \mathcal{A} satisfy:*

- $\Rightarrow Z$ is compact and infinite, $\mathcal{A} \sqsubseteq C(Z)$, and $\text{III}(\mathcal{A}) = Z$.
- $\Rightarrow V \subseteq Z$, V is clopen and non-empty, and $V \in \mathfrak{K}$.

then \mathcal{A} contains a non-trivial idempotent. Then, all spaces in \mathfrak{K} are removable.

Proof. Fix $K \in \mathfrak{K}$. Then fix X, U, \mathcal{A} satisfying the hypotheses of Definition 6.2. Let $Z = \text{III}(\mathcal{A})$. Assume that $Z \not\subseteq X \setminus U$. We shall derive a contradiction. Shrinking U , we may assume that U is clopen. Clearly $U \neq \emptyset$, so $|X| \geq 2$ (by $U \subsetneq X$), so X is infinite (by all idempotents trivial), so Z is infinite.

$\mathcal{A} \restriction Z \sqsubseteq C(Z)$ and $\text{III}(\mathcal{A} \restriction Z) = Z$. Let $V = Z \cap U$; then $V \neq \emptyset$. $V \in \mathfrak{K}$ because \mathfrak{K} is closed-hereditary. So, $\mathcal{A} \restriction Z$ contains a non-trivial idempotent, $f \restriction Z$, where $f \in \mathcal{A}$. But then $f^2 - f$ is 0 on Z and hence on X , so f is an idempotent, contradicting the hypotheses of Definition 6.2. ☕

Theorem 6.10 \mathfrak{R} is the class of all removable spaces.

Proof. Since \mathfrak{R} is clearly closed-hereditary, we may apply Lemma 6.9 to prove that all spaces in \mathfrak{R} are removable. Thus, assume that X is compact and infinite, $\mathcal{A} \sqsubseteq C(X)$, and $\text{III}(\mathcal{A}) = X$, and $V \subseteq X$ is clopen and non-empty, and $V \in \mathfrak{R}$. We must show that \mathcal{A} contains a non-trivial idempotent. We may assume that V is nowhere removable, and in particular perfect, since otherwise the result is clear from the definition of “removable”. Applying the definition of \mathfrak{R} , whenever U is a non-empty clopen subset of V , there is an $n_U \geq 2$ and a non-empty clopen H with $H \subseteq U$ and H both n_U -big and n_U -superdissipated. Taking a minimal n_U and shrinking V , we may assume that V itself is n -superdissipated, where $n \geq 2$,

and that whenever U is a non-empty clopen subset of V , there is a non-empty clopen H with $H \subseteq U$ and H n -big.

Since $X \setminus V$ is not a boundary, we may fix $\psi \in \mathcal{A}$ such that $\|\psi\| > 1$ but $|\psi(x)| \leq 1/2$ for all $x \notin V$. Then fix a non-empty clopen $H \subseteq V$ such that $|\psi(x)| \geq 1$ for all $x \in H$. Shrinking H , we may assume that H is n -big. We now get a non-trivial idempotent by Lemma 5.10. ☕

Corollary 6.11 *If $X \subseteq (I_S)^3$ is closed and does not contain a copy of the Cantor set, then X is removable.*

Proof. $X \in \mathfrak{R}$ by Lemmas 3.20, 5.21, 5.15, and 5.7. ☕

7 Powers of the Double Arrow Space

Here we show that arbitrary finite powers of the double arrow space $I_{(0,1)}$ are removable, and hence have the CSWP. This argument works because there is a certain uniformity in the standard map from $(I_{(0,1)})^k$ onto I^k , which is captured by the next definition:

Definition 7.1 *For $n \geq 1$, $\pi : X \rightarrow Y$ is n - \star tight iff for $y \in Y$ and $0 \leq j < n$, there are $K_y^j \subseteq X$ and $U_y^j \subseteq Y$ satisfying:*

1. X, Y are compact, Y is metric, and the Cantor set does not embed into X .
2. For each y : The K_y^j , for $j < n$, form a clopen partition of X , and each $|K_y^j \cap \pi^{-1}\{y\}| \leq 1$.
3. For each j : $\{(y, z) : z \in U_y^j\}$ is open in Y^2 .
4. For each y, j : $\pi^{-1}(U_y^j) \subseteq K_y^j$.
5. For each y, j : $K_y^j \setminus \pi^{-1}(U_y^j)$ is removable.

X is n - \star dissipated iff $\pi : X \rightarrow Y$ is n - \star tight for some π and Y .

Some of the K_y^j and U_y^j may be empty, so “ n - \star dissipated” get weaker as n gets bigger. Note that (2) implies that $|\pi^{-1}\{y\}| \leq n$ for each y , so that π is n -supertight by Lemma 3.24, and X is totally disconnected by Lemma 3.25. X is 1- \star dissipated iff X is compact and countable. The class of n - \star dissipated spaces is closed-hereditary, since if we have (1 – 5) and \tilde{X} is a closed subset of X , then we also have (1 – 5) for \tilde{X} , using $\pi|_{\tilde{X}} : \tilde{X} \rightarrow \tilde{Y} = \pi(\tilde{X})$, $\tilde{K}_y^j = K_y^j \cap \tilde{X}$, and $\tilde{U}_y^j = U_y^j \cap \tilde{Y}$.

Lemma 7.2 *If $(I_{(0,1)})^{k-1}$ is removable, then the standard map $\pi : (I_{(0,1)})^k \twoheadrightarrow I^k$ is 2^k - \star tight.*

Proof. As in Definition 5.18, let $\mathcal{D} = \{+, -\}^k$. For $y \in I^k$ and $\Delta \in \mathcal{D}$, let $U_y^\Delta = \{z \in I^k : y <^\Delta z\}$, and let $K_y^\Delta = \{t \in (I_{(0,1)})^k : y^\Delta \leq^\Delta t\}$. Then properties (1–4) are easily verified, and (5) holds because $K_y^j \setminus \pi^{-1}(U_y^j)$ is covered by finitely many homeomorphic copies of $(I_{(0,1)})^{k-1}$. ☕

We shall eventually prove:

Theorem 7.3 *If $n < \omega$ and X is n - \star dissipated, then X is removable.*

It follows that X is n - \star dissipated iff X is removable and there is a $\pi : X \twoheadrightarrow Y$ such that Y is compact metric and each $|\pi^{-1}\{y\}| \leq n$. To prove the \leftarrow direction: In Definition 7.1, take all $U_y^j = \emptyset$; the K_y^j may simply be chosen arbitrarily to satisfy condition (2). Thus, the notion of “ n - \star dissipated” becomes of little interest, but it was chosen to make the following proof work:

Proof of Theorem 1.6. Each $(I_{(0,1)})^k$ is in fact removable. This follows by induction on k , using Lemma 7.2 and Theorem 7.3. ☕

We shall now prove Theorem 7.3 by showing that $X \in \mathfrak{R}$ (see Definition 6.8).

Definition 7.4 *A compact space P is nowhere n - \star dissipated iff \overline{W} is not n - \star dissipated for all non-empty open $W \subseteq P$.*

Lemma 7.5 *If X is perfect and n - \star dissipated, then there is a non-empty clopen $V \subseteq X$ and an m with $2 \leq m \leq n$ such that V is m - \star dissipated and nowhere $(m-1)$ - \star dissipated.*

Theorem 7.3 will follow easily from the next two lemmas, about spaces which are n - \star dissipated and nowhere removable. Of course, the theorem implies that there are no such spaces.

Lemma 7.6 *Assume that $X, Y, n \geq 2, \pi$ and the K_y^j and U_y^j are as in Definition 7.1, with X nowhere $(n-1)$ - \star dissipated and nowhere removable.*

1. *For a fixed j and non-empty open $V \subseteq Y$: $U_y^j \cap V \neq \emptyset$ for some $y \in V$.*
2. *For any $\varepsilon > 0$, the sets:*

$$A_\varepsilon^j := \{z \in Y : \exists y[z \in U_y^j \text{ \& } d(y, z) < \varepsilon]\}$$

$$B_\varepsilon^j := \{y \in Y : \exists z[z \in U_y^j \text{ \& } d(y, z) < \varepsilon]\}$$

are dense and open in Y .

Proof. For (1): Assume that $U_y^j \cap V = \emptyset$ for all $y \in V$. Let W be a non-empty clopen subset of $\pi^{-1}\{V\}$, and consider the restriction $\pi|_W : W \rightarrow \tilde{Y} = \pi(W)$. $\tilde{U}_y^j = U_y^j \cap \tilde{Y} = \emptyset$ for each $y \in \tilde{Y}$ and $\tilde{K}_y^j = K_y^j \cap W = (K_y^j \setminus \pi^{-1}(U_y^j)) \cap W$ is empty for each $y \in \tilde{Y}$ because it is clopen in X and removable. But then, by deleting index j , we see that W is $(n-1)$ - \star dissipated; in the special case $n = 2$, W would be countable because X does not contain a Cantor subset.

For (2): They are open by (3) of Definition 7.1. If one of them fails to be dense, then there is a non-empty open $V \subseteq Y$ such that V is disjoint from either A_ε^j or B_ε^j . In either case, we may assume that $\text{diam}(V) < \varepsilon$ which implies that $z \notin U_y^j$ whenever $z, y \in V$, contradicting (1). ☕

Lemma 7.7 *If $n \geq 2$ and X is n - \star dissipated and nowhere $(n-1)$ - \star dissipated and nowhere removable, then X is n -big.*

Proof. Fix $\mathcal{A} \sqsubseteq C(X)$ and $\Upsilon : \mathcal{A} \rightarrow \omega$. We shall verify the conclusion of Definition 5.6 with $r = 1$, so we shall find $f_0, \dots, f_{n-1} \in \mathcal{A}$ and $c \in X$ such that the $\Upsilon(f_j)$, for $j = 0, \dots, n-1$, are all equal, and such that $|f_i(c) - f_j(c)| \geq 1$ whenever $0 \leq i < j < n$. Let Y , π and the K_y^j and U_y^j be as in Definition 7.1. Let $G = \bigcap \{A_\varepsilon^j \cap B_\varepsilon^j : \varepsilon > 0 \text{ \& } j < n\}$; by Lemma 7.6, $Y \setminus G$ is of first category in Y because the intersection may be taken just over rational ε .

If $y \in G$, then y is in the closure of each U_y^j , so that $\pi^{-1}\{y\}$ meets each K_y^j ; let x_y^j be the element of $\pi^{-1}\{y\} \cap K_y^j$. Since finite spaces have the CSWP, we may choose, for each $y \in G$, a $g_y \in \mathcal{A}$ such that $g_y(x_y^j) = 2j$ for each $j < n$. Then, chose a rational $\varepsilon_y > 0$ such that $|g_y(x) - 2j| < 1/2$ whenever $j < n$, $x \in K_y^j$, and $d(\pi(x), y) < \varepsilon_y$.

Now, fix $N \subseteq G$, $\varepsilon > 0$, and $\ell \in \omega$ such that N is not of first category in Y and $\varepsilon_y = \varepsilon$ and $\Upsilon(g_y) = \ell$ for all $y \in N$. Then, fix a point $d \in N$ and a δ with $0 < \delta < \varepsilon$ such that $N \cap B(d, \delta)$ is dense in $B(d, \delta)$. Let c be any point in $\pi^{-1}\{d\}$. For each $j < n$, $\{y : d \in U_y^j\}$ is open, and this set meets $B(d, \delta)$ (since $d \in N \subseteq A_\delta^j$), so choose $y^j \in N \cap B(d, \delta)$ such that $d \in U_{y^j}^j$, and we can let $f_j = g_{y^j}$; note that $d \in U_{y^j}^j \rightarrow c \in K_{y^j}^j \rightarrow |f_j(c) - 2j| < 1/2$. ☕

Proof of Theorem 7.3. Apply Theorem 6.10; every n - \star dissipated space X is in \mathfrak{R} by Lemmas 7.5 and 7.7. ☕

8 Remarks and Questions

Regarding our notion of bigness: From the point of view of general topology, the use of the “ $\mathcal{A} \sqsubseteq C(X)$ ” in Definition 5.6 seems a bit artificial, although it was

needed for the CSWP proofs. It would be more natural to restrict \mathcal{A} to be only $C(X)$, which would result in a weaker property; but we do not know if it would really be strictly weaker. Of course, we can always replace \mathcal{A} by $\text{cl}(\mathcal{A})$, so the two properties are equivalent when X has the CSWP.

The degree of bigness of some LOTSeS is easily calculated. Doing so lets us show (Example 8.5) that the “obvious” generalization of Lemma 5.15 is false. It is easy to see that $\omega_1 + 1$ is n -big for all n . But there is a class of LOTSeS for which the bigness is bounded. We do not state the most general possible result, but just say enough to verify Example 8.5, which uses the I_Λ from Definition 2.1.

Lemma 8.1 *Let $L = I_\Lambda$, where $\Lambda : I \rightarrow \omega$, and let K be any compact space which is not $(n + 1)$ -big. Let $X = L \times K$. Then X is not $(3n + 1)$ -big.*

Proof. Let $\sigma : L \rightarrow I$ be the standard map. Also, applying the definition of “not $(n + 1)$ -big”, fix $\mathcal{A} \subseteq C(K)$ and $\Upsilon : \mathcal{A} \rightarrow \omega$ such that for each $c \in K$ and each $f_0, f_1, \dots, f_n \in \mathcal{A}$ with $\Upsilon(f_0) = \Upsilon(f_1) = \dots = \Upsilon(f_n)$, there are $j < k \leq n$ such that $|f_j(c) - f_k(c)| < 1/4$.

Let $M = C(K)$, with the usual sup norm. For $f \in C(X)$, define $\tilde{f} \in C(L, M)$ by $(\tilde{f}(u))(z) = f(u, z)$. Let \mathcal{B} be the set of all $f \in C(X)$ such that $\tilde{f}(u) \in \mathcal{A}$ for all $u \in L$. Then $\mathcal{B} \subseteq C(X)$, and we shall define a partition Φ of \mathcal{B} into \aleph_0 pieces demonstrating that X is not $(3n + 1)$ -big. As a first approximation, for each $f \in \mathcal{B}$, choose $\Psi(f) = (m^f, \bar{y}^f, \bar{r}^f, \bar{s}^f, \bar{t}^f)$ so that:

1. $1 \leq m^f \in \omega$.
2. $\bar{y}^f = \langle y_i^f : 0 \leq i \leq 2m^f \rangle$, each $y_i^f \in I$, and $y_i^f \in \mathbb{Q}$ when i is even.
3. $0 = y_0^f < y_1^f < \dots < y_{2m^f}^f = 1$.
4. $\bar{r}^f = \langle r_i^f : 0 \leq i \leq 2m^f \rangle$, where each $r_i^f = |\sigma^{-1}\{y_i^f\}| - 1 = \Lambda(y_i^f)$.
5. $\|\tilde{f}(u) - \tilde{f}(v)\| \leq 1/4$ whenever $\max(\sigma^{-1}\{y_i^f\}) \leq u \leq v \leq \min(\sigma^{-1}\{y_{i+1}^f\})$.
6. $\bar{s}^f = \langle s_{i,\mu}^f : 0 \leq i \leq 2m^f \text{ \& } 0 \leq \mu \leq r_i^f \rangle$, where $\{s_{i,\mu}^f : 0 \leq \mu \leq r_i^f\} \subset L$ lists $\sigma^{-1}\{y_i^f\}$ in increasing order; so $s_{i,\mu}^f = (y_i^f, \mu)$.
7. $\bar{t}^f = \langle t_{i,\mu}^f : 0 \leq i \leq 2m^f \text{ \& } 0 \leq \mu \leq r_i^f \rangle$, where $t_{i,\mu}^f = \Upsilon(\tilde{f}(s_{i,\mu}^f))$.

Such a $\Psi(f)$ may be chosen using compactness, plus continuity of \tilde{f} . Of course, there are 2^{\aleph_0} possible values of $\Psi(f)$ because of the y_i^f and $s_{i,\mu}^f$ for odd i , so we delete these and define $\Phi(f) = (m^f, \bar{y}^f, \bar{r}^f, \bar{s}^f, \bar{t}^f)$, where

8. $\bar{y}^f = \langle y_i^f : 0 \leq i \leq 2m^f \text{ \& } i \text{ is even} \rangle$.
9. $\bar{s}^f = \langle s_{i,\mu}^f : 0 \leq i \leq 2m^f \text{ \& } i \text{ is even \& } 1 \leq \mu \leq r_i^f \rangle$.

There are only countably many possible values for $\Phi(f)$, so if X were $(3n+1)$ -big, we could fix a $(b, c) \in X = L \times K$ and $f_0, \dots, f_{3n} \in \mathcal{A}$ such that the $\Phi(f_j)$ are all the same, and such that $|f_j(b, c) - f_k(b, c)| \geq 1$ whenever $j < k \leq n$. We shall now derive a contradiction. Write $\Phi(f_j) = (m, \vec{y}, \vec{r}, \vec{s}, \vec{t})$.

If $b = s_{i,\mu}^f$ for some even i , then the $\Upsilon(\tilde{f}_j(b)) = t_{i,\mu}^f$ are all the same, and we contradict our assumptions on Υ just using $\tilde{f}_0(b), \dots, \tilde{f}_n(b)$. So, we may fix an even $i < 2m$ so that $\max(\sigma^{-1}\{y_i\}) < b < \min(\sigma^{-1}\{y_{i+2}\})$. Now, for each $j \in \{0, 1, \dots, 3n\}$, there are three cases:

- I. $\max(\sigma^{-1}\{y_i\}) < b < \min(\sigma^{-1}\{y_{i+1}^{f_j}\})$.
- II. $b \in \sigma^{-1}\{y_{i+1}^{f_j}\}$.
- III. $\max(\sigma^{-1}\{y_{i+1}^{f_j}\}) < b < \min(\sigma^{-1}\{y_{i+2}\})$.

So, one of these cases must happen for $n+1$ values of j . We shall assume that this is Case I, since the argument is essentially the same in the other two cases. Permuting the f_j , we may assume that Case I holds for $0 \leq j \leq n$. Fix $\mu = r_i$, so that $\max(\sigma^{-1}\{y_i\}) = s_{i,\mu}$, so $\Upsilon(\tilde{f}_j(s_{i,\mu})) = t_{i,\mu}^f$ for each $j \leq n$. By our assumptions on Υ , we may fix $j < k \leq n$ such that $|f_j(s_{i,\mu}, c) - f_k(s_{i,\mu}, c)| < 1/4$. Applying Condition (5) above, we have $|f_j(b, c) - f_k(b, c)| < 3/4$, contradicting $|f_j(b, c) - f_k(b, c)| \geq 1$. ☹

In particular, letting K be the 1-point space, we see that an I_Λ is not 4-big. Then, proceeding by induction,

Lemma 8.2 $\prod_{j < m} L_{\Lambda_j}$ is not $(3^m + 1)$ -big.

We remark that in the proof of Lemma 8.1, we could have replaced the “ $<$ ” by “ \leq ” in Cases I and III, although then they would not be disjoint from Case II. However, in the special case of $L = I_S$, Case II can now be eliminated, so that we can replace the “ $(3n+1)$ -big” by “ $(2n+1)$ -big”, obtaining:

Lemma 8.3 Let $L = I_S$, where $S \subseteq I$, and let K be any compact space which is not $(n+1)$ -big. Let $X = L \times K$. Then X is not $(2n+1)$ -big.

Lemma 8.4 $\prod_{j < m} I_{S_j}$ is not $(2^m + 1)$ -big.

Example 8.5 For any $n > 3$, there is an X which does not have a Cantor subset and which is not 7-big, such that X is $(n+2)$ -superdissipated and not $(n+1)$ -superdissipated.

Proof. For $n \geq 1$, let $L_n = I_{\Lambda_n}$, where $\Lambda_n(x) = n$ for $n \in (0, 1)$, and $\Lambda_n(0) = \Lambda_n(1) = 0$. Then L_1 is the double arrow space. Let $X = L_n \times L_1$. Then X is not 7-big by Lemmas 8.2 and 8.3. X is $(n+2)$ -dissipated by Lemma 3.4. To prove that X is not $(n+1)$ -dissipated, it is sufficient (by Lemma 3.6 of [12]) to observe that for each $\varphi \in C(L_n, [0, 1]^\omega)$ there is a $z \in [0, 1]^\omega$ with $|\varphi^{-1}\{z\}| \geq n+1$. ☕

It is easily seen using Lemma 5.20 that $(I_S)^n$ is 2^n -big when S is uncountable, so it has the NTIP, since it is also $(2^{n-1} + 1)$ -superdissipated. However, it is not clear whether it has the CSWP in the case that S meets all Cantor sets, since the natural proof requires looking at arbitrary perfect subspaces of $(I_S)^n$.

Example 8.6 *If $S \subseteq (0, 1)$ meets all Cantor sets, then there is a perfect $X \subseteq (I_S)^4$ such that X is not 8-dissipated, is not 7-big, and has no Cantor subsets.*

Proof. Let $D \subset (I_S)^3$ be the diagonal, and let $X = D \times I_S$. Then D is the same as the LOTS obtained from I by replacing each point in S by eight points. Since $7 = 2 \cdot 3 + 1$, Lemmas 8.2 and 8.3 show that X is not 7-big. The proof of Example 8.5 shows that X is not 8-dissipated. ☕

This particular X has the CSWP, and in fact is removable (see Section 6), since I_S is removable, and after removing the clopen copies of I_S from X , we are left with a copy of $(I_S)^2$, which is also removable. We do not know whether $(I_S)^4$ itself must have the CSWP.

A simple example of n -big spaces is given by:

Proposition 8.7 *If X is compact and $|X| > 2^{\aleph_0}$, then X is n -big for all $n \in \omega$.*

Proof. Fix $\mathcal{A} \subseteq C(X)$, fix $n \in \omega$, and fix $\Upsilon : \mathcal{A} \rightarrow \omega$. We shall verify the conclusion of Definition 5.6 with $r = 1$.

Let \mathbb{P} be the set of all finite partial functions from X to ω ; so each $p \in \mathbb{P}$ is a function with $\text{dom}(p)$ a finite subset of X and $\text{ran}(p) \subseteq \omega$. For $p \in \mathbb{P}$, choose an $f_p \in \mathcal{A}$ with $p \subset f_p$.

For each $c \in X$ and $s \in \omega$, let $E_{c,s} = \{p(c) : p \in \mathbb{P} \text{ \& } c \in \text{dom}(p) \text{ \& } \Upsilon(f_p) = s\} \subseteq \omega$. If some $|E_{c,s}| \geq n$ then we are done, so assume that $|E_{c,s}| \leq n-1$ for all c, s . There are only 2^{\aleph_0} possibilities for $\langle E_{c,s} : s \in \omega \rangle$, so we can fix an infinite $A \subseteq X$ and sets $E_s \in [\omega]^{<n}$ for $s \in \omega$ such that $E_{c,s} = E_s$ for all $c \in A$ and all $s \in \omega$. But then $\text{ran}(p) \subseteq E_s$ whenever $p \in \mathbb{P}$ and $\Upsilon(f_p) = s$ and $\text{dom}(p) \subseteq A$. Now choosing p with $\text{dom}(p) \subseteq A$ and $|\text{ran}(p)| = n$ yields a contradiction. ☕

Finally, the following Ramsey-type lemma might be of interest for studying products of LOTSEs, although we never needed it in this paper. The proof uses the terminology from Definition 5.18.

Proposition 8.8 *Fix an uncountable $J \subseteq \mathbb{R}^n$, and assume that*

$$\forall x, y \in J [x \neq y \rightarrow \forall i < n [x_i \neq y_i]] \quad (*)$$

Then there is a 1-1 function $\varphi : \mathbb{Q} \rightarrow J$ such that for all $i < n$:

$$\forall p, q [p < q \rightarrow \varphi(p)_i < \varphi(q)_i] \quad \text{or} \quad \forall p, q [p < q \rightarrow \varphi(p)_i > \varphi(q)_i] \quad (\dagger)$$

Proof. Call a box $B = \text{box}[a, b]$ *big* iff $B \cap J$ is uncountable; By $(*)$, this implies that $B^\circ \cap J$ is uncountable, where B° denotes the interior of B . For $\Delta \in \mathcal{D}$, let $-\Delta$ result from interchanging the signs $+$ and $-$ in Δ . Call the box B Δ -*bad* iff B is big and there is no $d \in B^\circ \cap J$ such that $\text{corn}(d, B, \Delta)$ and $\text{corn}(d, B, -\Delta)$ are both big. Observe, for any big box B :

1. B is Δ -bad iff B is $(-\Delta)$ -bad.
2. If B is Δ -bad and $A \subseteq B$ is a big box, then A is Δ -bad.
3. There is some $\Delta \in \mathcal{D}$ such that B is not Δ -bad.

(1) and (2) are obvious. To prove (3), we note first that if we replaced \mathbb{Q} by ω or a finite set in the statement of the lemma, then the result would be obvious by Ramsey's Theorem. Now, let Z be the set of points of $B^\circ \cap J$ which are condensation points of J . Obtain $\varphi : \{0, 1, 2\} \rightarrow Z$ so that (\dagger) holds replacing \mathbb{Q} by $\{0, 1, 2\}$. Let $a = \varphi(0)$, $b = \varphi(2)$, and $d = \varphi(1)$. By (\dagger) , there is some $\Delta \in \mathcal{D}$ such that $a \in \text{corn}(d, B, \Delta)$ and $b \in \text{corn}(d, B, -\Delta)$, and then $\text{corn}(d, B, \Delta)$ and $\text{corn}(d, B, -\Delta)$ are both big.

Using (2) (sub-boxes go from bad to worse) and (3), we can fix a $\Delta \in \mathcal{D}$ and a big box B such that for all big boxes $A \subseteq B$, A is not Δ -bad. We may now list \mathbb{Q} in type ω and obtain φ in ω steps. When $p < q$, we shall have $\varphi(p)_i < \varphi(q)_i$ when $\Delta_i = +1$ and $\varphi(p)_i > \varphi(q)_i$ when $\Delta_i = -1$. ☕

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